The Helen of Geometry

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The seventeenth century is one of the most exciting periods in the history of mathematics. The first half of the century saw the invention of analytic geometry and the discovery of new methods for finding tangents, areas, and volumes. These results set the stage for the development of the calculus during the second half. One curve played a central role in this drama and was used by nearly every mathematician of the time as an example for demonstrating new techniques. That curve was the cycloid.

The cycloid is the curve traced out by a point on the circumference of a circle, called the *generating circle*, which rolls along a straight line without slipping (see Figure 1). It has been called it the "Helen of Geometry," not just because of its many beautiful properties but also for the conflicts it engendered.

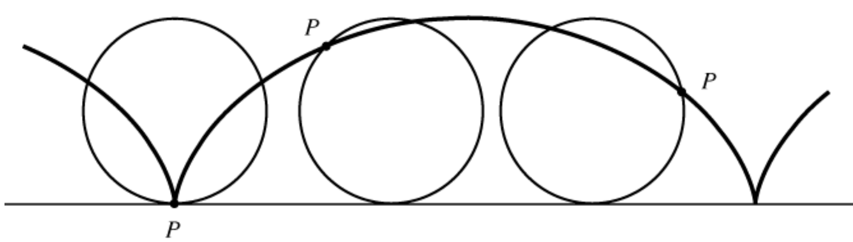

Figure 1. The cycloid.

This article recounts the history of the cycloid, showing how it inspired a generation of great mathematicians to create some outstanding mathematics. This is also a story of how pride, pettiness, and jealousy led to bitter disagreements among those men.

Early history

Since the wheel was invented around 3000 B.C., it seems that the cycloid might have been discovered at an early date. There is no evidence that this was the case. The earliest mention of a curve generated by a point on a moving circle appears in 1501, when Charles de Bouvelles [**7**] used such a curve in his mechanical solution to the problem of squaring the circle.

Galileo was one of the first to give serious consideration to the cycloid. In a letter to Cavalieri in 1640, he said that he had been thinking about the curve for more than fifty years. It was Galileo who named it the 'cycloid,' from a Greek word meaning circle-like. He also attempted its quadrature, that is, finding the area of the region under one arch of the curve. His method was simple and direct. He cut the arch and generating circle from the same material and compared their weights, concluding that the area under an arch is close to three times that of the generating circle [**15**].

Father Marin Mersenne (after whom Mersenne primes are named) became interested in the cycloid around 1615. He was in contact with many prominent mathematicians and it is possible that he had heard of the curve from Galileo. Mersenne noted that the distance along the baseline between consecutive cusps of the cycloid is equal to the circumference of the generating circle. (One advantage of being the first to consider an idea is that the early theorems are fairly obvious!) He thought that the curve itself might be a semiellipse [**1**, p. 156]. This guess is not too far off, as Figure 2 shows.

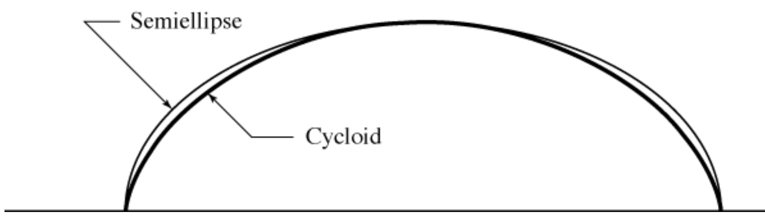

Figure 2. A comparison of the cycloid and semiellipse.

In 1628, Mersenne encouraged Gilles de Roberval to study the cycloid. By 1634, Roberval had solved the quadrature problem, showing that the area under one arch of the curve is *exactly* three times the area of the generating circle. By 1638, he could draw the tangent to the curve at any point, and had found the volume generated when the region under an arch is revolved about the baseline. He chose not to publish these results, which led to some unpleasant priority disputes. For whenever someone announced a theorem on the cycloid, Roberval would send out letters claiming to have found the same result earlier. His reluctance to publish may be explained by the fact that he held the Ramus chair of mathematics in the Collège Royal, a position filled every three years by an open competition. The incumbent was responsible for writing the problems used in the competition, so it was to his advantage to keep his best methods secret. Roberval did this well. His methods were not published until 1693, eighteen years after his death [**1**, p. 150].

Roberval's representation

From our modern perspective, the most efficient way to represent the cycloid is in parametric form, using trigonometric functions. In Roberval's time, the values of the trigonometric ratios were well known, but the function concept had not been developed. He studied the cycloid as a curve defined in the following way (see Figure 3) [**6**, p. 447]: "Let the diameter $\overline{AB}$ of the circle *AGB* move along the tangent $\overline{AC}$, always remaining parallel to its original position until it takes the position $\overline{CD}$, and let AC be equal to the semicircle *AGB*. At the same time, let the point *A* move on the semicircle *AGB*, in such a way that the speed of $\overline{AB}$ along $\overline{AC}$ may be equal to the speed of *A* along the semicircle *AGB*. [In this way, the distance from A to X along the baseline will be equal to the distance from X to A' along the semicircle, as it would be if the circle were rolling.] Then, when $\overline{AB}$ has reached the position $\overline{CD}$, the point A will have reached the position D. Thus the point A is carried along by two motions—its own on the semicircle *AGB*, and that of the diameter along $\overline{AC}$."

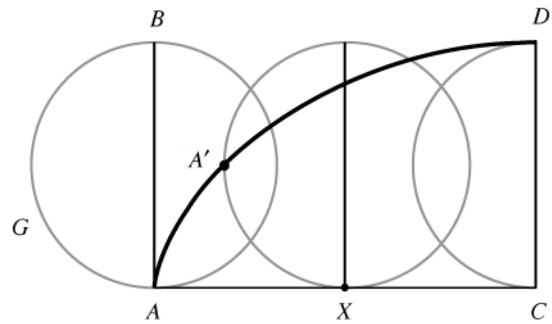

Figure 3. Roberval's definition of the cycloid.

Roberval's quadrature

To find the area of the region under one arch of the cycloid Roberval began by drawing a new curve, which he called the *companion curve*, constructed in the following way. Let P be any point on the cycloid. Along a line parallel to $\overline{AC}$, draw $\overline{PQ}$ congruent to the semi-chord $\overline{EF}$. (See Figure 4 where additional semi-chords and their corresponding segments are shown.) The companion curve is the locus of Q. The curve is actually a sinusoid though Roberval was probably unaware of this.

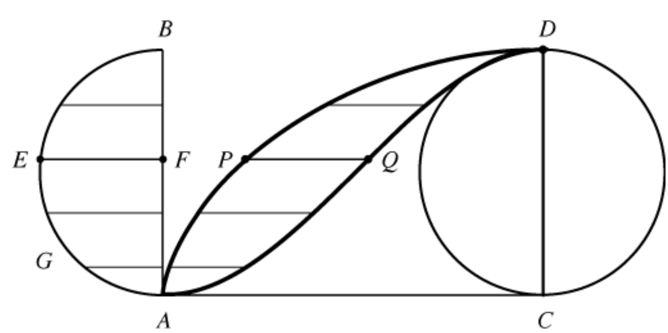

Figure 4. The companion curve to the cycloid.

Roberval showed that the area of the region between the cycloid and its companion curve is half the area of the generating circle, or $\frac{1}{2}\pi a^2$. This follows from the two-dimensional version of Cavalieri's principle: "If two regions bounded by parallel lines are such that any parallel between them cuts each region in segments of equal length, then the regions have equal area." Since the companion curve is constructed so that slices through both regions at equal heights have the same length, the areas are equal. Roberval used Cavalieri's principle again to show that the companion curve divides the rectangle *ABDC* into two parts with equal areas (see Figure 5), since for each segment $\overline{MN}$ in the region on the left there is a corresponding congruent segment $\overline{RS}$ in the region on the right. Hence, the area under the companion curve is equal to half the area of the rectangle, or $\frac{1}{2}\left[\frac{1}{2}(2\pi a) \times 2a\right] = \pi a^2$ Now, the area under one arch of the cycloid is twice the sum of the areas of regions *APDR* and *AQDC*. It follows that this area is $3\pi a^2$ as promised.

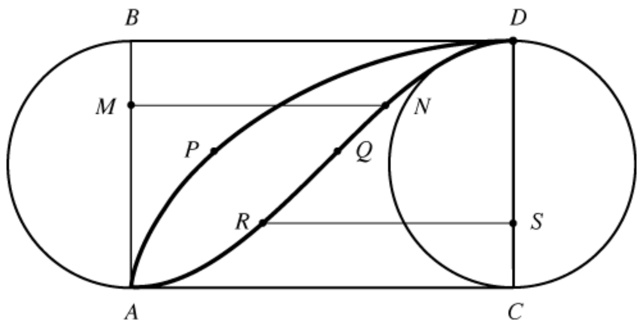

Figure 5. The companion curve bisects the rectangle *ABDC*.

Descartes' tangent construction

Roberval sent his quadrature to Mersenne who in turn challenged Rene Descartes and Pierre de Fermat to verify the result, which they did. When Descartes learned how Roberval had calculated the area he commented [**11**], "Roberval has labored overmuch to produce so small a result." In response, Roberval replied, "Prior knowledge of the answer to be found has no doubt been of assistance." In August of 1638 Roberval, Fermat, and Descartes each gave Mersenne a different method for drawing tangents to the cycloid. In the argument that followed between Fermat and Descartes over who had the better construction, Descartes denounced Fermat's demonstration as "the most ridiculous gibberish I've ever seen." Roberval sided with Fermat and this prompted Descartes to write letters to Mersenne criticizing Roberval's construction. This quarrel was due in part to the fact that there was no general agreement at the time on what it meant for a line to be tangent to a curve [**15**]. It was Fermat's approach that eventually led to the modern definition of the tangent as the limiting position of the secant.

Here is Descartes' ingenious solution. To draw the tangent at any point P on the half arch of the cycloid *AGD* (see Figure 6), first draw $\overline{PE}$ parallel to the base $\overline{AC}$ intersecting the circle at E. Next draw $\overline{PQ}$ parallel to $\overline{EC}$ and $\overleftrightarrow{PH}$ perpendicular to $\overline{PQ}$ at P. $\overleftrightarrow{PH}$ is the required tangent line [**15**].

Descartes gave the following justification. In place of a rolling circle consider a rolling polygon; in the case of Figure 7, a hexagon *ABCDEF*. As the hexagon rolls along the line, vertex A will trace out a sequence of circular arcs whose centers will be at the points B, C', D', and so on. The tangents to each of these arcs will be perpendicular to the line joining the point of tangency to the center of the arc. Descartes [**2**, p. 309] argued, "the same would happen with a polygon of one hundred thousand million sides, and consequently also with the circle." His construction determines point Q in Figure 6, the location of the generating circle along the baseline that corresponds to point P.

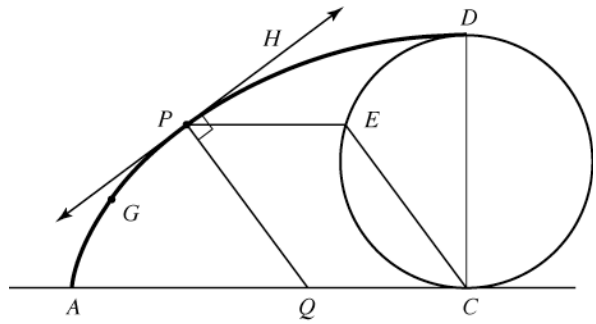

Figure 6. Descartes' tangent construction.

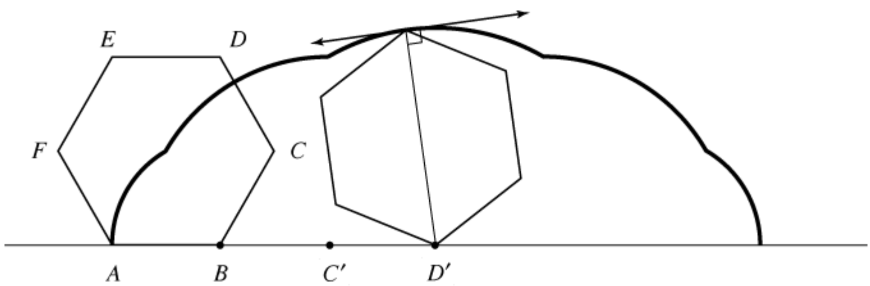

Figure 7. Justification of Descartes' construction.

Torricelli's publication

In 1644, Evangelista Torricelli [**13**, p. 410], a pupil of Galileo, published his quadrature of the cycloid in an appendix to his book *Opera Geometrica*, where he wrote: "One now asks what proportion the cycloidal space has to its generating circle. We demonstrate (and may thanks be given to God) that it is triple." Torricelli's treatise is the first known article on the cycloid. Roberval was furious with Torricelli for publishing a result that he considered to be his own discovery and in a widely circulated letter, accused him of stealing the proof. Torricelli died in 1647 shortly after learning of this charge of plagiarism. A rumor started that his death was the result of the shame he felt at being accused of such a dishonorable deed. The truth is that he died from typhoid as he was attempting to gather material to support the originality of his work. It is now known that both men made their discoveries about the cycloid, independently of one another [**8**].

Pascal's contest

After demonstrating mathematical talent at an early age, Blaise Pascal turned his attention to theology, denouncing the study of mathematics as a vainglorious pursuit. Then one night, unable to sleep as the result of a toothache, he began thinking about the cycloid and to his surprise, his tooth stopped aching. Taking this as a sign that he had God's approval to continue, Pascal spent the next eight days studying the curve. During this time he discovered nearly all of the geometric properties of the cycloid. He issued some of his results in 1658 in the form of a contest, offering a prize of forty Spanish gold pieces and a second prize of twenty pieces. There were two main problems [**10**, p. 343]:

1. Determine the area and the center of gravity of that part of half a cycloidal arch, which is above a line parallel to the baseline.
2. Determine the volume and center of gravity of the solid generated when the region from problem (1) is revolved about its base and also about its vertical boundary.

Only two papers were received and neither was judged good enough to win the contest. One came from the British mathematician John Wallis, who later corrected a number of errors in his solution and published it. Pascal published his solutions, along with an essay on the "History of the Cycloid." This pamphlet caused a great deal of resentment among the Italians since Pascal took Roberval's side in his dispute with Torricelli.

Wren's arc length calculation

Until the 1650's the problem of rectification, that is, finding a straight line equal in length to the arc of a curve, was thought by many to be unsolvable. In fact Descartes [**1**, p. 163], in his second book on geometry, wrote that the relation between curved lines and straight lines is not, nor ever can, be known. By this he meant that it would not be possible to express the length of a curve as a rational multiple of the length of a line segment. However, at the time of Pascal's contest, Sir Christopher Wren, the famous English architect, sent Pascal his claim that the length of the cycloidal arch is *exactly* eight times the radius of the generating circle. Wren did not include a proof, though he had one. This result was new to Pascal, but Roberval insisted that he had proved it some years earlier. Wallis published Wren's proof, which used infinite series, in his treatise on the cycloid.

It is important to note that the problems we have discussed so far—Roberval's quadrature, Descartes' tangent construction, Pascal's contest problems, and Wren's rectification—were first solved geometrically using only the simplest of infinite processes. These original solutions required considerable thought and extraordinary insight. With the invention of the calculus, they became simple exercises.

Huygens' tautochrone

In 1657, the Dutch mathematician, Christiaan Huygens, was thinking about how to make a more accurate clock. Mersenne suggested using a pendulum as a timing device, but Huygens knew that the period of a circular pendulum is not independent of its amplitude and wrote [**16**, p. 71], "In a simple pendulum the swings that are elongated more from the perpendicular are slower than the others. And so in order to correct this defect at first I suspended the pendulum between two curved plates . . . , which by experiment I learned in what way and how to bend in order to equalize the larger and smaller swings."

What Huygens did was to place nails in the path of a pendulum made with a flexible cord (see Figure 8). The nails altered the path of the bob so that it followed a sequence of circular arcs. By trial and error he was able to construct a system whose period was independent of its amplitude. To force the bob to travel along a smooth rather than piecewise path, he replaced the nails with a pair of curved plates. Inaccuracies in his design caused him to abandon the curved plates in favor of the simpler method of having the pendulum swing in small circular arcs. It is this type of clock that is described in his first book on the subject, *Horologium*, published in 1658.

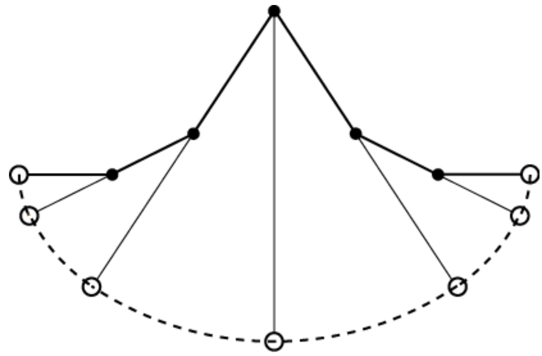

Figure 8. Huygens' pendulum.

His abandonment of the curved plates was short lived. Inspired by Pascal's contest, he noticed that the bob of his curved plate pendulum appeared to follow a cycloid. He was able to show that the frequency of an object forced to follow an inverted cycloid is independent of its amplitude. Thus the cycloid is the *tautochrone* (from the Greek *tauto*, same + *chronos*, time); the curve for which the time taken by a particle, freely accelerated by gravity, to reach the lowest point on the curve is the same regardless of its starting point. Huygens published this result in 1673, in his book *Horologium Oscillatorium*. Here he described his design of a clock made with a flexible pendulum that swung between two curved plates that force the bob to travel on a cycloidal path. Constructed in this way, the pendulum will maintain the same period, regardless of the amplitude of the swing.

Huygens' demonstration that the cycloid is the tautochrone, is a geometric one that follows a long and difficult route [**16**, p. 50]. Using only first year-calculus, the demonstration is straightforward. See for example [**14**].

The involute

In 1659, Huygens attacked the problem of calculating the shape of the curved plates mathematically. In so doing he discovered another remarkable property of the cycloid. Consider Figure 9, which shows two consecutive arches of a cycloid. Imagine that a thread, attached at point A, is wrapped around the curve to point B at the top of the arch. As the tightly stretched thread is unwound, its end traces out what is known as an *involute*. What surprised Huygens was the fact that this involute of the cycloid is another cycloid of the same size! Thus, the curved plates were also cycloids, and he could calculate their precise shape. Unfortunately, the cycloidal pendulum clock was

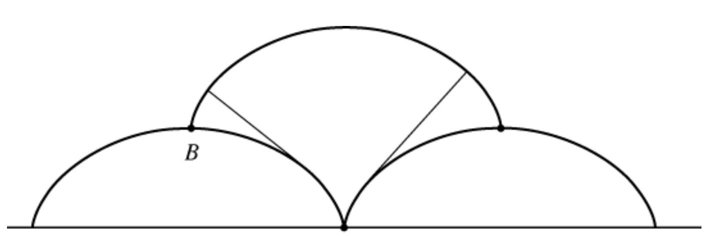

Figure 9. An involute of the cycloid.

not the success that Huygens had hoped because in practice, friction made it no more accurate than a circular one.

Bernoulli's brachistochrone

In the June 1696 issue of *Acta Eruditorum* [**6**, p. 497] the following problem was proposed: "If two points *A* and *B* are given in a vertical plane, to assign to a mobile particle *M* the path *AMB* along which, descending under its own weight, it passes from the point *A* to the point *B* in the briefest time." The author was Johann Bernoulli and the problem became known as the *brachistochrone* (from the Greek *brachistos*, shortest + *chronos*, time). Certainly, the shortest distance between *A* and *B* lies along the line segment (a) that joins them (see Figure 10). However, the goal is to find the curve that minimizes the time of descent. It is not obvious which curve accomplishes this. Could it be an arc of a parabola (b), or a circle (c), or something a little more radical (d)?

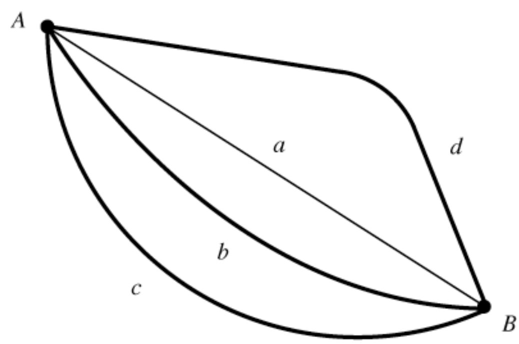

Figure 10. The brachistochrone problem.

Before the problem was published, Johann, his brother Jacob, and Leibniz had solved it, finding that the quickest descent occurred along an arc of a cycloid. They decided to issue it as a challenge to other mathematicians. In the public announcement Bernoulli declared [**9**], " . . . there are fewer who are likely to solve our excellent problems, aye, fewer even among the very mathematicians who boast that they have wonderfully extended its bounds by means of the golden theorems which (they thought) were known to no one, but which in fact had long previously been published by others." This was a thinly veiled attack on Newton, who by this time had left Cambridge and was Warden of the Mint. It is likely that the Bernoullis and Leibniz hoped to embarrass Newton by showing that he was unable to solve their problem. Legend has it that Newton received the problem in the mail one afternoon and had solved it by morning. He published his solution anonymously in the *Philosophical Transactions of the Royal Society* in January of 1697 [**9**]. A charming, but perhaps apocryphal, story claims that on seeing the solution Bernoulli at once exclaimed, "Ah! I recognize the lion by his paw." The May 1697 issue of the *Acta* contained solutions to the problem by Newton, Leibniz, and the Bernoullis. Newton was pleased that he had solved the problem, but wrote later [**4**, p. 201], "I do not love to be dunned and teased by foreigners about Mathematical things." In the end, Bernoulli grudgingly acknowledged Newton's talents and Leibniz, who was embarrassed by the incident, wrote to the Royal Society denying any part in it.

In his published solution, Newton did not actually prove that the cycloidal path gives the shortest time, but instead showed how to construct the cycloid joining any two points. Given points A and B in a vertical plane, what is needed is the radius of the circle that when rolled along the horizontal line $\overleftrightarrow{AC}$, will generate a cycloid starting at A and passing through B (see Figure 11). Newton drew an arbitrary cycloid starting at A and intersecting line $\overleftrightarrow{AB}$ at D. He noted that since all cycloids are similar, it must be that AD is to AB as the radius of the given circle is to that of the unknown circle. From this proportion the radius of the generating circle of the solution curve can be found [**9**].

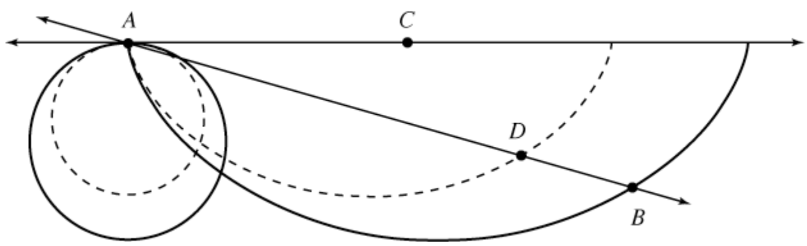

Figure 11. Newton's solution.

The two Bernoullis found quite different solutions, and the brothers feuded over priority. Each accused the other of plagiarism. Johann began his demonstration with the words " . . . the reader will be greatly amazed when I say that exactly this cycloid, or tautochrone of Huygens, is our required brachistochrone." His solution is a clever blend of physics and mathematics. He found the problem equivalent to that of determining the path followed by a ray of light as it travels through a transparent medium of steadily decreasing density.

Bernoulli's reasoning proceeds as follows. Begin by assuming that the light passes through a medium consisting of several layers, each with its own uniform density, which decreases as we move down through the layers (see Figure 12). As the ray passes from one layer to the next, its direction changes and its speed increases. Fermat's principle of least time states that as light crosses a boundary between layers, it takes the quickest route from one point to another. Since the light is speeding up, as an object would if it were falling, it seems reasonable that the solution to the brachistochrone might be found by imitating the behavior of light. Note that as the number of layers increases without bound, the path becomes a smooth curve. This analysis led Johann to a differential equation whose solution is the cycloid [**12**, p. 392].

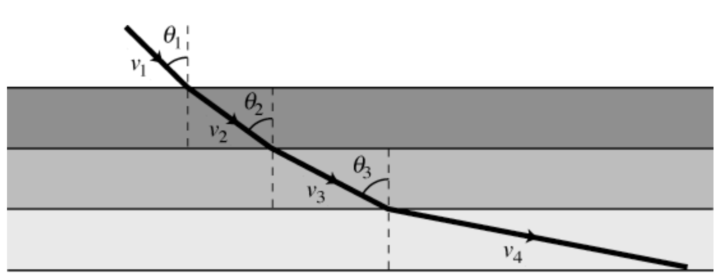

Figure 12. The path of light thorough a medium of variable density.

He concluded his demonstration with the words, " ... I must voice once more the admiration I feel for the unexpected identity of Huygens' tautochrone, and my brachistochrone ... Nature always tends to act in the simplest way, and so it here lets one curve serve two different functions." He was so proud of his discovery that he had inscribed on the title page of his *Collected Works*, the drawing of a dog admiring the figure of a cycloid. A banner over the picture proclaims the Latin motto, "*Supra Invidiam*"—above envy.

Johann's solution to the brachistochrone is clever, but his optical analogy could not be generalized. On the other hand, Jacob's solution attacked the "variable curve" aspect of the problem. That is, the cycloid must be singled out from all possible curves between A and B. In modern notation his method amounts to determining the function $y(x)$ that minimizes the integral

$$\frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1 + [y'(x)]^2}{y(x)}}\, dx,$$

which represents the time of descent. Jakob's solution was the first step in the development of the calculus of variations. (See [**3**] and [**5**] for two recent articles in this journal that present modern solutions to the brachistochrone problem.)

Concluding remarks

By the end of the seventeenth century, mathematicians had discovered all of the secrets of the cycloid and turned their attention to other curves. It often happens in the history of mathematics that a certain idea or problem will appear at precisely the right time. This was the case with the cycloid. The discoveries of its beautiful geometric and mechanical properties are closely connected with the history of analytic geometry and the calculus. The challenges and battles that were fought over it led to significant advances. No other curve could have served the same purpose. Certainly, the cycloid has earned the title "The Helen of Geometry."

Summary. The cycloid has been called the Helen of Geometry, not only because of its beautiful properties but also because of the quarrels it provoked between famous mathematicians of the 17th century. This article surveys the history of the cycloid and its importance in the development of the calculus.

Acknowledgments. The author wishes to thank the editors and anonymous referees for their valuable advice and encouragement.

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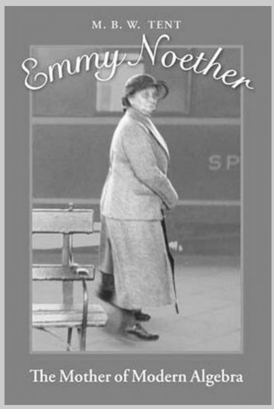
Additional Resources

The following websites provide animations of the curves and problems discussed in this article.

The Cycloid—http://demonstrations.wolfram.com/CycloidCurves/
The Tautochrone—http://www.mathwords.com/t/tautochrone.htm
The Brachistochrone—http://home.imm.uran.ru/iagsoft/brach/BrachJ2.html

Emmy Noether?



"Emmy Noether: The Mother of Modern Algebra," a biography of the famous algebraist intended for young adult readers, is reviewed on page 72. The cover photo (shown above on the right) is controversial. Several reviewers (ours among them) deplore this choice of image and question whether it is indeed a photo of Emmy Noether at all. The cover photo comes from the Oberwolfach Photo Collection (http://owpdb.mfo.de/). The well-known photo on the left is from the Universitätsarchiv Göttingen. Since publication of the book, the cover photo has been removed from the Oberwolfach web site. It was donated to Oberwolfach by Peter Roquette, who was given it by Margot Chow, widow of the mathematician W. L. Chow, who studied with Noether at Göttingen in 1932. Mrs. Chow, who has since died, judged that Emmy Noether was the subject by the position of the photo amongst her husband's papers. What do you think?

Proof Without Words: Area of a Cycloidal Arch

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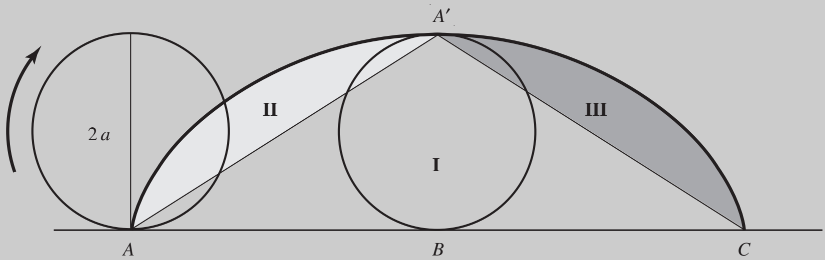
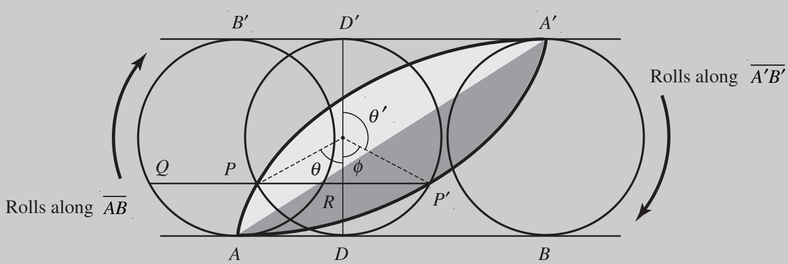


$$\text{Area(under arch)} = \mathbf{I} + \mathbf{II} + \mathbf{III} = \frac{1}{2}(AC \cdot A'B) + \mathbf{II} + \mathbf{III}$$

$$= \frac{1}{2}(2\pi a \cdot 2a) + \mathbf{II} + \mathbf{III} = 2\pi a^2 + \mathbf{II} + \mathbf{III}$$



$$a\theta + a\theta' = \widehat{PD} + \widehat{P'D'} = AD + A'D' = \pi a$$

$$\Rightarrow \theta + \theta' = \pi \Rightarrow \theta = \phi \Rightarrow \overline{AB} \parallel \overline{PP'} \Rightarrow QR = PP'$$

$$\Rightarrow \text{Area(circle } AQB'R) = \text{Area(lens } APA'P') \quad \text{[Cavalieri's principle]}$$

$$\Rightarrow \pi a^2 = \text{Area}(APA'P') = \mathbf{II} + \mathbf{III}$$

Thus, Area(under arch) $= 2\pi a^2 + \pi a^2 = 3\pi a^2$.

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